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PROPERTIES SUPERSOLUBLE GROUPS

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Abstract

A metabelian group is a group G that possesses a normal_subgroup N such that N and G/N are both abelian. Equivalently, G is metabelian if and only if the commutator subgroup [G,G] is abelian. Equivalently again, G is metabelian if and only if it is solvable of length at most 2

Let G=AB be the mutually permutable product of the supersoluble subgroups A and B.

Then G/F (G)is supersoluble.

Keywords: mutually permutable product- totally permutable, supersoluble subgroups

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1. Introduction.

It is known that a group which is the product of two super soluble groups is not necessarily super soluble, even if the two factors are normal subgroups of the group. Baer proved in [3] that if a group G is the product of two normal supersoluble groups and G′ is nilpotent, then G is super soluble. The search for generalizations of Baer's result has been a fruitful topic of investigation recently (see [3,4]).

subgroups A and B such that $A\cap B=1$, then the product is in fact totally permutable [6, Proposition 3.5], and therefore G is supersoluble. Our main Theorem is a generalization of this last property.

Theorem 1.

Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. If $Core_G(A\cap B)=1$, then G is supersoluble.

Lemma 1[4, Theorem 2]. If G=AB is the mutually permutable product of the supersoluble subgroups A and B, then G is soluble.

Lemma 2. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then, either G is supersoluble or $NA < G$ and $NB < G$ for every minimal normal subgroup N of G.

Proof. Assume that G is not supersoluble. Then both A and B are proper subgroups of G. Let N be a minimal normal subgroup of G and for contradiction assume that NA=G. Then, as N is abelian, $N\cap A$ is a normal subgroup of

$\langle N,A\rangle = G$. Since N is a minimal normal subgroup of G and $A<G$, we have that $N\cap A=1$ and consequently A is a maximal subgroup of G. Clearly we can also assume that B is not contained in A. It is not difficult to argue that we can choose an element b of $B\backslash A$ such that $bq\in A$ for some prime q. Since the product G=AB is mutually permutable, $A\langle b\rangle$ is a subgroup of G and the maximality of A implies that $G=A\langle b\rangle$. We now take orders to reach our final contradiction:

$|A||N|=|G|=|A||\langle b\rangle||A\cap\langle b\rangle|=q|A|$. Consequently we have that |N|=q and then G is supersoluble, a contradiction.

Lemma 3. Let G=AB be the mutually permutable product of the subgroups A and B and let N be any minimal normal subgroup of G. Then either

$N\cap A=N\cap B=1$ or $N=(N\cap A)(N\cap B)$.

Proof. Let N be a minimal normal subgroup of G. Clearly $A(N\cap B)$ and $(N\cap A)B$ are both subgroups of G. Note that A normalizes $N\cap(A(N\cap B))=(N\cap A)(N\cap B)$ and B normalizes $N\cap((A\cap N)B)=(N\cap A)(N\cap B)$. Therefore $(N\cap A)(N\cap B)$ is a normal subgroup of G and the minimality of N yields the result.

Lemma 4. Let G be a group, and N a minimal normal subgroup of G such that |N|=pn, where p is a prime and n>1. Denote C=CG(N) and assume that G/Cis supersoluble. Then, if Q/Cis a subgroup of G/C containing Op′(G/C), we have that Q is normal in G and N=∏ti=1Ni, where Ni are non-cyclic minimal normal subgroups of NQ for i=1,...,t.

Proof. Since by [8, Lemma A.13.6], we have that Op(G/C)=1 and the commutator subgroup (G/C)′ of G/C is nilpotent because e G/C is supersoluble, it follows that (G/C)′is a p′-group. Therefore (G/C)′ is contained in Op′(G/C) and thus Op′(G/C) is a Hall p′-subgroup of G/C. Consequently, Q/Cis a normal subgroup of G/C and hence Q is normal in G. Consider now N as a G-module over GF (p)by conjugation. Then, by Clifford's Theorem [8, Theorem B.7.3], N viewed as a Q-module is a direct sum N=∏ti=1Ni, where Ni are irreducible Q-modules for i=1,...,t. Suppose that there exists i∈{1,...,t}such that |Ni|=p. Then clearly |Nj|= p for all j. Therefore Q/CQ(Ni) is abelian of exponent dividing p−1, and the same is true for Q/C. In particular, Q/C=Op′(G/C) is a Hall p′-subgroup of G/C. Since N is not cyclic, it follows that Q = G and thus p divides |G/C|. Hence p is the largest prime dividing |G/C|. From the supersolubility of G/C, we obtain that 1= Op(G/C) is a Sylow subgroup of G/C, a contradiction. Consequently, Ni is a non-cyclic minimal normal subgroup of NQ for all i∈{1,...,t}, as we wanted to prove.

Proof of Theorem 1. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B, with CoreG(A∩B)=1, and suppose that G has been chosen minimal such that its supersoluble residual GU is non-trivial. Let N be a minimal normal subgroup of G contained in GU. Note that N is an elementary abelian p-group for some prime p. Applying Lemma 2, we have that both NA and NB are proper subgroups of G. Moreover, using Lemma 3, we have that either N=(N∩A)(N∩B) or N∩A=N∩B=1. Assume first that N=(N∩A)(N∩B).

(i) If N∩A=1, then N is cyclic. Assume that N∩A=1. It follows that N is contained in B. Let N0 be a non-trivial cyclic subgroup of N. Since AN0 is a subgroup of G, we have that N0 =AN0∩N is anormal subgroup of AN0. Hence every cyclic subgroup of N is normalised by A. Now let N1 be a minimal normal subgroup of B contained in N. Since B is supersoluble, it follows

That N1 is cyclic and thus normalised by A. Hence N1 is a normal subgroup of G. The minimality of N implies that N=N1 and consequently N is cyclic.

(ii) N∩A=1 and N∩B=1.On the contrary, assume that N∩A=1. From (i), we know that N is cyclic. Moreover, Nis contained in B. Hence AN∩B= (A∩B)N. Let L=CoreG(A∩B)N). Clearly, N is contained in Land L=L∩((A∩B)N)=(L∩A∩B)N. It is clear that G/L=(AL/L)(BL/L)is a mutually permutable product of AL/L and BL/L such that CoreG/L((AL/L)∩(BL/L))=1. By the minimality of G, it follows that G/L is supersoluble. On the other hand, since N is cyclic, we have that G/CG(N) is abelian. Hence G/CL(N) is supersoluble and GUCL(N)=C. Note that C=N×(C∩A∩B).

Therefore $C \cap A \cap B$ contains a Hall p'-subgroup of C. Since $Core_G(A \cap B)=1$ and $O_{p'}(C)$ is a normal subgroup of G contained in $C \cap A \cap B$, we have that $O_{p'}(C)=1$. Moreover, $C'=(C \cap A \cap B)'$ is a normal subgroup of G contained in $A \cap B$. Consequently, $C'=1$ and C is an abelian p-group. In particular, GU is abelian and thus GU is complemented in G by a supersoluble normalizer D which is also a supersoluble projector of G, by [8, Theorems V.4.2 and V.5.18]. Since N is cyclic, we know that N is central with respect to the saturated formation of all supersoluble groups. By [8,Theorem V.3.2.e], Dcovers N and thus N is contained in D. It follows $ND \cap GU=1$, a contradiction.

(iii) Either $N=N \cap A$ or $N=N \cap B$. If we have $N=N \cap A=N \cap B$, then N is contained in $A \cap B$, contradicting the fact that $Core_G(A \cap B)=1$. We may assume without loss of generality that $N \cap A=N$.

(iv) AN and BN are both supersoluble. Since $N=(N \cap A)(N \cap B)$ and $N=N \cap A$, it follows that $N \cap B$ is not contained in $N \cap A$. Let n be any element of $N \cap B$ such that $n \in N \cap A$, and write $N0 = \langle n \rangle$. Note that AN0 is a subgroup of G, and $AN0 \cap N=(N \cap A)N0$. Therefore $N0(N \cap A)$ is a normal subgroup of AN0, and consequently A normalizes $(A \cap N)N0$. This yields that $A/C_A(N/N \cap A)$ acts as a power automorphism group on $N/N \cap A$. This means that AN is supersoluble. If $N \cap B=N$, then $BN=B$ is supersoluble. On the contrary, if $N \cap B=N$, we can argue as above and we obtain that BN is supersoluble. Consequently, $ACG(N)/CG(N)$ and $BCG(N)/CG(N)$ are both abelian groups of exponent dividing $p-1$. But then $G/CG(N)=(ACG(N)/CG(N))(BCG(N)/CG(N))$ is a π-group for some set of primes π such that if $q \in \pi$, then q divides $p-1$.

(v) Let B0 be a Hall π-subgroup of B. Then $AB0 \cap N= A \cap N$.

This follows just by observing that $AB0 \cap N$is contained in each Hall π'-subgroup of AB0 and every Hall π'-subgroup of A is a Hall π'-subgroup of AB0. Note that $|G/CG(N)|$ is a π-number and AB0 contains a Hall π-subgroup of G. Therefore $G=(AB0)CG(N)$. But then $A \cap N$ is a normal subgroup of G. The minimality of G yields either $A \cap N=$ 1or $A \cap N= N$. This contradicts our assumption $1=N \cap A=N$, and so we cannot have $N=(A \cap N)(B \cap N)$. Thus, by Lemma 3, we may assume $N \cap A=N \cap B=1$. Let $M= Core_G(AN \cap BN)$. Then $N \cap M= N$ and G/M is supersoluble by the minimality of G. Again, we reach a contradiction after several steps.

(vi) M=N. Suppose that M=N. Since G/M is supersoluble, we know that N cannot be cyclic. Let us write $C=CG(N)$, and consider the quotient group G/C. It is clear that G/C is supersoluble. Let $Q/C=O_p(G/C)$. Since $O_p(G/C)=1$ and $(G/C)'$is nilpotent, it follows that Q/C is a normal Hall p'-subgroup of G/C. Let $B_{p'}$ be a Hall p'-subgroup of B. Since $|N|$ divides $|B:A \cap B|$, we have that $(A \cap B)B_{p'}$ is a proper subgroup of B. Let T be a maximal subgroup of B containing $(A \cap B)B_{p'}$. Then AT is a maximal subgroup of G and $|G:AT|= p= |B:T|$. If N is not contained in AT, we have $G=(AT)N$

and AT∩N=1. Then |N|=p, a contradiction. Therefore N is contained in AT. In particular, the family S={X:X is a proper subgroup of B, (A∩B)Bp′X and NAX} is non-empty. Let R be an element of S of minimal order. Observe that AR has p-power index in G and thus ARC/C contains Op′(G/C). Regarding N as a AR-module over GF (p), we know, by Lemma 4, that N is a direct sum N=∏ti=1Ni, where Ni is an irreducible AR-module whose dimension is greater than 1, for all i∈{1,...,t}. Assume that (A∩B)Bp′=R. Then AR=ABp′ and thus N is contained in A, a contradiction. Therefore ABp′∩B=(A∩B)Bp′ is a proper subgroup of R. Let S be a maximal subgroup of R containing (A∩B)Bp′. From the minimality of R, we know that N is not contained in AS. Consequently, there exists some i∈{1,...,t} such that Ni is not contained in AS, which is a maximal subgroup of AR. Hence AR=(AS)Ni. Since Ni is a minimal normal subgroup of AR, it follows that AS∩Ni= 1and |Ni|= |AR:AS|= |R:S|= p, a contradiction.

(vii) M is an elementary abelian p-group. Note that M=N(M∩A)=N(M∩B) and |M∩A|=|M∩B|=|M|/|N|. Moreover, A(M∩B)is a subgroup of G such that A(M∩B)∩M= (M∩A)(M∩B). Hence (M∩A)(M∩B) is also a subgroup of G. If M∩A= M∩B, then M∩A is a normal subgroup of G contained in A∩B. This implies that M∩A=1 and consequently M=N, a contradiction. It yields that M∩A=M∩B. Next we see that (M∩A)(M∩B) is a normal subgroup of G. Since (M∩A)(M∩B)= M∩A(M∩B), we have that A normalizes (M∩A)(M∩B). Similarly, B normalises

(M∩A)(M∩B) since (M∩A)(M∩B)= M∩B(M∩A). This implies normality of (M∩A)(M∩B) in G. Let X=(M∩A)(M∩B). Since we cannot have M∩A= M∩B, M∩A must be strictly contained in X. Thus X=X∩M=(X∩N)(M∩A) > M∩A gives us X∩N=1. But then X∩N=N, giving NX. Suppose that Q is a Hall p′-subgroup of M∩B. Then QA is a subgroup and so QA∩M=Q(M∩A) is also a subgroup which contains Q. Hence, as |M:M∩A|=pk for some k, we have that QM∩A∩B. Thus QB∩MM∩A∩B and similarly QA∩MM∩A∩B. Consequently, QM is contained in M∩A∩B. Since QM=Op(M), it follows that Op(M) is a normal subgroup of G contained in A∩B. Hence Op(M)=1, a contradiction, and consequently Q=1andMis a p-group. Hence N is contained in Z(M) and M=N×(M∩A)=N×(M∩B). Thus φ(M)=φ(M∩A)=φ(M∩B) is a normal subgroup of G contained in A∩B. This implies that φ(M)=1 and M is an elementary abelian p-group, as claimed.

(viii) Final contradiction. We have from the previous steps that M∩A is not contained in M∩B and that M∩B is not contained in M∩A because otherwise, since |M∩A|=|M∩B|, it follows that M∩A=M∩B is a normal subgroup of G contained in A∩B. This would imply M∩A=M∩B=1, and M=(M∩A)N=N. This fact contradicts step (vi).

Let x be an element of M∩B such that x/∈M∩A. Then A 〈x〉 is a subgroup of G, and so is M0=A 〈x〉 ∩M=(A∩M) 〈x〉 . Therefore M0 is an A-invariant subgroup of G. In particular, since M=(M∩A)(M∩B), we have that every subgroup of M/M∩A is A-invariant; that is, A/CA(M/M∩A) acts as a group of power automorphisms on M/M∩A. It is clear that M/M∩A is A-isomorphic to N.

Consequently, A/CA(N) acts as a group of power automorphisms on N. This implies that A normalises each subgroup of N. Analogously, B normalises each subgroup of N. It follows that N is a cyclic group. We argue as in step (ii) above to reach a final contradiction. We have that G/M is supersoluble and M is abelian. Therefore GUM and thus GU is abelian and complemented in G by a supersoluble normaliser, D say, by [8, Theorem V.5.18]. Since N is cyclic, we know that D covers N and thus NGU∩D=1, a contradiction. Proof of Theorem 2.

Let M=GU denote the supersoluble residual of G. Theorem 1yields that G/CoreG(A∩B) is supersoluble. Therefore M is contained in CoreG(A∩B). In particular, M is supersoluble. Let F(M) be the Fitting subgroup of M. Since A and Bare supersoluble, we have that [M,A]F(A)∩MF(M) and [M,B]F(B)∩MF(M). Consequently, [M,G] is contained in F(M). Note now that the chief factors of G between F(M) and Mare cyclic, and recall that G/M is supersoluble. Therefore we have that G/F (M) is supersoluble. This implies that M=F(M) and thus M is nilpotent. Consequently, G/F (G) is supersoluble. We now show that G/F (G) is metabelian. We prove first that A′ and B′ both centralise every chief factor of G. Let H/K be a chief factor of G. If H/K is cyclic, then as G′ centralizes H/K, so do A′ an dB′. Hence we may assume that H/K is a non-cyclic p-chief factor of G for some prime p. Note that we may assume that H is contained in M because G/M is supersoluble and H/K is non-cyclic. To simplify notation, we can consider K=1. Since F(G) centralizes H [8, Theorem A.13.8.b], G/CG(H) is supersoluble. Let Ap′ be a Hall p′-subgroup of A. By Maschke's theorem [8, Theorem A.11.5], H is a completely reducible Ap′-module and HAp′ is supersoluble because H is contained in A. Therefore Ap′/CAp′(H) is abelian of exponent dividing p−1. This implies that the primes involved in |A/CA(H)| can only be p or divisors of p−1.The same is true for |B/CB(H)|. This implies that if p divides |G/CG(H)|, then p is the largest prime dividing |G/CG(H)|. But since Op(G/CG(H))=1 and G/CG(H) is supersoluble, it follows that G/CG(H) must be a p′-group. Consider H as A-module over GF (p). Since ACG(H)/CG(H) is a p′-group, we have that H is a completely reducible A-module and every irreducible A-submodule of H is cyclic. Consequently A′ centralizes H, and the same is true for B′. Let now U/V be a chief factor of G. Then G/CG(U/V)is the product of the abelian subgroups ACG(U/V)/CG(U/V) and BCG(U/V)/CG(U/V). By Itô's theorem [9], we have that G/CG(U/V)is metabelian. Since F(G)is the intersection of the centralisers of all chief factors (again by [8, Theorem A.13.8.b]), we can conclude that G/F (G) is metabelian.3. Final remarks Finally, Theorem 1 enables us to give succinct proofs of earlier results on mutually permutable products. is a cyclic group as well. This implies that G is supersoluble, a contradiction.

Main Theorem. Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then G/F (G)is supersoluble and metabelian.

Proof: by Theorem 1

Notice: This last theorem cannot be improved easily, as the following example shows.

Example. Let S3 be the symmetric group of degree 3, given by

S3=⟨α, β:α2=β3=1;βα=β2⟩ and let T7 be the non-abelian group of order 73 and exponent 7. Write T7= ⟨a,b⟩ with a7=b7=[a,b]7=1 and let c=[a,b]. We have that S3 acts on T7 in the following way: aα=b, bα=a, cα=c−1, aβ=a2, bβ=b4, cβ=c. Thus we can consider the semidirect product G=[T7] S3. Take now the subgroups

A= T7 ⟨β⟩ and B=T7 ⟨α⟩ of G. Clearly both A and B are supersoluble, and it is easy to check that G=AB is the mutually permutable product of A and B. Finally, we show that Theorem 1 provides elementary proofs for the results of Asaad and Shaalan about mutually permutable products.2.

Main results The following four lemmas are needed to prove Theorem 1.

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